5. MOMENTS

The general first-order moments $\langle \mathcal{E}^{\beta} \rangle$, $(0 \leq \beta < \infty)$, are now easily obtained from the results of the preceeding Section. Since

$$\left\langle \mathcal{E}^{\beta} \right\rangle \equiv \int_0^{\infty} w_1(\mathcal{E})\mathcal{E}^{\beta} d\mathcal{E}, \qquad 0 \leq \beta < \infty \quad , \tag{5.1}$$

(β real and nonnegative), we may apply (4.2) for Class A interference, and (4.5), with (4.3), (4.4), for Class B noise, respectively.

5.1 Existence and Direct Calculation (Approximate Forms):

For Class A interference we get directly

$$\begin{aligned} \left\langle \mathcal{E}^{\beta} \right\rangle_A &\cong e^{-A_A} \sum_{m=0}^{\infty} (2\hat{\sigma}_{mA}^2)^{\beta/2} \Gamma(\tfrac{\beta}{2}+1) A_A^m/m! \\ &= e^{-A_A} \Gamma(\tfrac{\beta}{2}+1) \sum_{m=0}^{\infty} \left(\frac{m/A_A + \Gamma_A'}{1+\Gamma'} \right)^{\beta/2} \frac{A_A^m}{m!} \quad , \end{aligned} \tag{5.2}$$

cf. (3.5). The sum in (5.2) is clearly finite, since by Stirling's theorem ($m! \simeq m^m e^{-m}\sqrt{2\pi m}$) for sufficiently large m ($\gg A_A \Gamma_A'$) the summand is dominated by $A_A^m / m^{m+1-\beta}$. Accordingly, all (finite) moments* exist for Class A interference, and are given by (5.2), approximately. [We recall that we consider only the principal development of P_{1A}, w_{1A}, cf. (3.7), (4.2) above.] Typical moments here are, from (5.2):

$$\left\langle \mathcal{E}^{0} \right\rangle_A = 1 \quad , \text{ as required;}$$

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* Of the envelope, and of the instantaneous amplitude [cf. Middleton, 1974, Section 4.5], by the same argument.

$$\langle\mathcal{E}\rangle_A = e^{-A_A}\ \frac{\sqrt{\pi}}{2}\ \sum_{m=0}^{\infty}\left(\frac{m/A_A+\Gamma'_A}{1+\Gamma'_A}\right)^{1/2}\frac{A_A^m}{m!}\quad ;$$

$$\langle\mathcal{E}^2\rangle_A \cong e^{-A_A}\ \sum_{m=0}^{\infty}\left(\frac{m/A_A+\Gamma'_A}{1+\Gamma'_A}\right)\frac{A_A^m}{m!} \quad = 1,\ \text{(as required)}. \tag{5.3}$$

For the Class B interference we have, from (4.3), (4.4) in (5.1):

$$\langle\mathcal{E}^\beta\rangle_B = \int_0^{\mathcal{E}_B}\mathcal{E}^\beta w_1(\mathcal{E})_{B-I}\,d\mathcal{E} + \int_{\mathcal{E}_B}^{\infty}\mathcal{E}^\beta w_1(\mathcal{E})_{B-II}\,d\mathcal{E} \tag{5.4a}$$

$$\cong 2\sum_{n=o}^{\infty}\frac{(-1)^n}{n!}\ \hat{A}_\alpha^n\Gamma(1+n\alpha/2)\int_0^{\mathcal{E}_B}\hat{\mathcal{E}}\,\mathcal{E}^\beta\ {}_1F_1(1+n\alpha/2;1;-\hat{\mathcal{E}}^2)\,d\mathcal{E}$$

$$+\frac{e^{-A_B}}{4G_B^2}\sum_{m=0}^{\infty}\frac{A_B^m}{m!}\int_{\mathcal{E}_B}^{\infty}\mathcal{E}^{\beta+1}e^{-\mathcal{E}_B^2/2\hat{\sigma}_{mB}^2}\,d\mathcal{E}/\hat{\sigma}_{mB}^2\ . \tag{5.4b}$$

[The integration of the hypergeometric function directly from its series form yields

$$\frac{N_I}{2G_B}\int_0^{\mathcal{E}_B}\mathcal{E}^{\beta+1}\,{}_1F_1(1+n\alpha/2;1;-\hat{\mathcal{E}}^2)\,d\mathcal{E} = \sum_{k=0}^{\infty}\frac{(-1)^k\mathcal{E}_B^{\beta+2k+2}(1+\alpha n/2)_k}{(k!)^2(\beta+2k-2)(2G_B)^{2k}}\left(\frac{N_I^{2k+1}}{2G_B}\right) \tag{5.5a}$$

which is probably the most direct and convenient form for numerical integration here.] In addition, the second integral in (5.4b) may be expressed as an incomplete gamma function, I_c, which is tabulated [K. Pearson, 1951], e.g.

$$\int_{\mathcal{E}_B}^{\infty}\mathcal{E}^{\beta+1}e^{-\mathcal{E}^2/2\hat{\sigma}_{mB}^2}\,d\mathcal{E}/\hat{\sigma}_{mB}^2 = (2\hat{\sigma}_{mB}^2)^{\beta/2}\left\{1-I_c[\mathcal{E}_B^2/2\hat{\sigma}_{mB}^2;1+\beta/2]\right\}\ , \tag{5.5b}$$

where I_C is defined by

$$I_C(x;\gamma) \equiv \frac{1}{\Gamma(\gamma)} \int_0^x y^{\gamma-1} e^{-y} dy \ . \tag{5.5c}$$

Again, direct integration of the integral itself in (5.5b) is probably the most convenient numerical procedure.] Clearly, by the same argument used above for the Class A noise [cf. (5.2) et seq.], all (finite) moments exist for Class B interference, as well. Some typical moments here are, from (5.4):

$$\left\langle \mathcal{E}^o \right\rangle_B = 1, \text{ as required;} \tag{5.6a}$$

$$\left\langle \mathcal{E} \right\rangle_B \simeq 2 \sum_{n=o}^{\infty} \frac{(-1)^n}{n!} \hat{A}_\alpha^n \, \Gamma(1+n\alpha/2) \int_0^{\mathcal{E}_B} \hat{\mathcal{E}}^{\beta} \mathcal{E} \ {}_1F_1(1+n\alpha/2;1;-\hat{\mathcal{E}}^2) d\mathcal{E}$$
$$+ \frac{e^{-A_B}}{4G_B^2} \sum_{m=0}^{\infty} (2\hat{\sigma}_{mB}^2)^{\frac{\beta+1}{2}} \frac{A_B^m}{m!} \left\{ 1 - I_C[\mathcal{E}_B^2/2\hat{\sigma}_{mB}^2; 3/2+\beta/2] \right\} \ ; \tag{5.6b}$$

$$\left\langle \mathcal{E}^2 \right\rangle_B \simeq 2 \sum_{n=o}^{\infty} \frac{(-1)^n}{n!} \hat{A}_\alpha^n \Gamma(1+n\alpha/2) \int_0^{\mathcal{E}_B} \hat{\mathcal{E}}^{\beta} \mathcal{E}^2 \ {}_1F_1(1+n\alpha/2;1;-\hat{\mathcal{E}}^2) d\mathcal{E}$$
$$+ \frac{e^{-A_B}}{4G_B^2} \sum_{m=0}^{\infty} (2\hat{\sigma}_{mB}^2)^{\beta/2+1} \frac{A_B^m}{m!} \left\{ 1 - I_C[\mathcal{E}_B^2/2\hat{\sigma}_{mB}^2; 2+\beta/2] \right\} < \infty \ ; \tag{5.6c}$$

[For numerical calculation we may also evaluate the integrals (left member of (5.5b)) directly, replacing β by $\beta+1$, $\beta+2$, respectively for the mean and mean square, (5.6b), (5.6c).]

As required from physical considerations, e.g. finite energy in the Class B (and A) interference, the second moment, in particular, is itself finite. This is not the case if we use the approximate forms $w_1(\mathcal{E})_{B-I}$, or $w_1'(\mathcal{E})_{B-I}$ $(=-dP_{1(B-I)}'/d\mathcal{E}$, cf. (3.21) et seq.), for all values of the envelope. Then, we have, in effect

$$\left\langle \mathcal{E}^{\beta'} \right\rangle \le \int_0^{\mathcal{E}_B} \mathcal{E}^{\beta} \hat{w}_1(\hat{\mathcal{E}})_{B-I} d\mathcal{E} + \frac{2\Gamma(1+\alpha/2)}{\Gamma(-\alpha/2)} A_\alpha \int_{\mathcal{E}_B}^{\infty} \mathcal{E}^{\beta-\alpha-1} d\mathcal{E} \ , \tag{5.7}$$

this last term from (3.31). The second term of (5.7) is finite only if $0 \leq \beta < \alpha$ (<2). Accordingly, no β moments exist for the Class (B-I) distributions, unless β is less than the spatial density - propagation parameter α. Thus, $\langle \mathcal{E}^2 \rangle_{B-I} \to \infty$, in contradiction to the physical situations we are attempting to model. In some cases (e.g. atmospheric noise) w_{B-I}, P_{B-I} are quite satisfactory for even very large values of $\mathcal{E}$, cf. Figs. (3.3) II, (3.4) II, but there is always some finite $\mathcal{E}_B$ beyond which a suitable form of w_{B-II}, or P_{B-II} must be used, in keeping with the bounded nature of all the moments.

5.2 Class A and B Moments: Exact Forms (Even Moments Only):

As we have seen above in Sec. (5.1) all (first-order) Class A and B moments of the envelope (and hence of the instantaneous amplitude) exist and are given, an approximate form, by Eqs. (5.2), (5.4). These results are approximate, albeit good ones.

However, an alternative development is possible, which can provide us with exact expressions, for the even-order moments.* For this we use (2.24b) and take its $[(-i)^k \, d^k/dr^k]$ derivative at r=0, to get

$$\left[(i)^k \frac{d^k}{dr^k} \hat{F}_1(ir)\right]_{r=0} = \left\langle E^k \cos^k\psi \right\rangle_{E,\psi} = \left\langle E^k \right\rangle \left\langle \cos^k\psi \right\rangle \; , \; k = 0,1,2,\ldots, \tag{5.8}$$

this last, since $W_1(E,\psi) = W_1(E)W_1(\psi)$, cf. (2.20), (2.21), a result of the narrow-band nature of the output from the (aperture x RF x IF) stages of the receiver. Since $W_1(\psi)$ is uniform in $(0,2\pi)$, all k-odd moments of the phase vanish, e.g. $\langle \cos^k\psi \rangle = 0$; $k = 1,3,5,\ldots$, and only the k-even moments remain. With the help of

$$\left\langle \cos^{2k}\psi \right\rangle_\psi = {}_{2k}C_k/2^{2k} = (2k)!/2^{2k}(k!)^2 \tag{5.9}$$

[Middleton, 1960, Eq. (5.26)] we accordingly obtain from (5.8) the general relation

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* No such results are available for the odd-order cases; one must use the approximate forms (5.2), (5.4).

$$\boxed{\left\langle E^{2k}\right\rangle = \frac{(k!)^2 2^{2k}}{(2k)!}\left[(-1)^k \frac{d^{2k}}{dr^{2k}}\hat{F}_1(ir)\right]_{r=0}} \tag{5.10}$$

for the even moments, <u>when they exist</u> (as they do here, but see the comments below in Section 5.3). In normalized form [with the aid of (3.3)] we can write (5.10) equivalently as

$$\boxed{\left\langle \varepsilon^{2k}\right\rangle = \frac{(k!)^2 2^{2k}}{(2k)!}\left[(-1)^k \frac{d^{2k}}{d\lambda^{2k}}\hat{F}_1(ia\lambda)\right]_{\lambda=0}\ .} \tag{5.10a}$$

Our general result (5.10), (5.10a) is equivalent to the procedures used by Furutsu and Ishida [1960, Section 6] and Giordano [1970, Appendix II, p. 175 et seq.], which is derived in our study by a different process.

<u>A. Exact Class A Even Moments:</u>

For Class A, even-order moments, including an independent gaussian component, the <u>exact</u> form is now obtained by using (2.50) in (5.10) or (5.10a), rather than from the approximate relations (2.77), (2.78). The simplest procedure here is to expand the c.f. $\hat{F}_1(ia\lambda)_A$ as a power series in λ^2, which is permitted, since the integral (in the exponent) is a definite integral, continuous in λ^2. The functional <u>form</u> of $\hat{F}_1(ia\lambda)_A$ is seen to be precisely that of the c.f. $F_1(i\xi)_{A=P+G}$ derived in the earlier study for the statistics of the instantaneous amplitudes (Class A noise), [Middleton, 1974, Section 4]. Accordingly, we use that expansion of F_1 to write at once here (exactly)

$$\hat{F}_1(ia\lambda)_A = 1 - \frac{a^2\lambda^2}{2!}\Omega_{2A}(1+\Gamma_A') + \frac{a^4\lambda^4}{4!}\left[\frac{3}{2}\Omega_{4A} + 3\Omega_{2A}^2(1+\Gamma_A')^2\right]$$
$$- \frac{a^6\lambda^6}{6!}\left[\frac{5}{2}\Omega_{6A} + \frac{45}{2}\Omega_{4A}\Omega_{2A}(1+\Gamma_A') + 15\Omega_{2A}^3(1+\Gamma_A')^3\right] + 0(a^8\lambda^8), \tag{5.11a}$$
$$= 1 - \frac{1}{2!}\frac{\lambda^2}{2} + \frac{\lambda^4}{4!}\left[\frac{3\Omega_{4A}}{8\Omega_{2A}^2(1+\Gamma_A')^2} + \frac{3}{4}\right] - \frac{\lambda^6}{6!}\left[\frac{5\Omega_{6A}}{16\Omega_2^3(1+\Gamma_A')^3} + \frac{45\Omega_{4A}}{16\Omega_{2k}^2(1+\Gamma_A')^2} + \frac{15}{8}\right]$$
$$+ \ \ldots\ldots, \tag{5.11b}$$

where we have

$$\Omega_{2k-A} \equiv A_A \left\langle \hat{B}_{oA}^{2k} \right\rangle / 2^k, \text{ (cf. (2.75d))} \; ; \quad \Gamma'_A \equiv \sigma_G^2 / \Omega_{2A} \text{ , cf. (3.1a),}$$

with

$$\langle \; \rangle \equiv \left\langle \int_0^{z_0} (\,) dz \right\rangle_{[\theta = z_0, A_0, e_{o\gamma}, \mathcal{a}_{RT}]} \text{ , cf. (2.64c), (2.65), (2.75d) for } \left\langle \hat{B}_{oA}^{2k} \right\rangle .$$

Applying (5.11) to (5.10a) and observing that the expression in the square brackets [ ] in (5.10a) is precisely the coefficient of $(-1)^k (a\lambda)^{2k}/(2k)!$ in (5.11a) [or of $(-1)^k \lambda^{2k}/(2k)!$ in (5.11b)], we obtain

$$[\langle \mathcal{E}^0 \rangle_A = 1] \text{ (as expected) ;} \tag{5.12a}$$

$$\langle \mathcal{E}^2 \rangle_A = 2a^2 \Omega_{2A}(1+\Gamma'_A) = 1 \text{ (as expected) ;} \tag{5.12b}$$

$$\langle \mathcal{E}^4 \rangle_A = \frac{8}{3} a^4 [\tfrac{3}{2} \Omega_{4A} + 3\Omega_{2A}^2 (1+\Gamma'_A)^2] = \frac{\Omega_{4A}}{\Omega_{2A}^2 (1+\Gamma'_A)^2} + 2 \; ; \tag{5.12c}$$

$$\langle \mathcal{E}^6 \rangle_A = \frac{16a^6}{5} [\tfrac{5}{2} \Omega_{6A} + \tfrac{45}{2} \Omega_{4A} \Omega_{2A} (1+\Gamma'_A) + 15\Omega_{2A}^3 (1+\Gamma'_A)^3]$$

$$= \frac{\Omega_{6A}}{\Omega_{2A}^3 (1+\Gamma'_A)^3} + \frac{9\Omega_{4A}}{\Omega_{2A}^2 (1+\Gamma'_A)^2} + 6 \text{ ; etc.} \tag{5.12d}$$

(This is given in <u>un</u>normalized form, e.g. with $\mathcal{E}/a = E$, by (5.12) on replacing $\mathcal{E}$ by E and deleting <u>a</u> therein.) These results (5.12) are to be compared with the approximate (and sometimes exact) forms (5.3). For example, observe that when $\Gamma'_A \to 0, \infty$, $\langle \mathcal{E}^2 \rangle = 1$, as required, cf. (5.12b), with similar equivalences for $\langle \mathcal{E}^4 \rangle$, etc., from (5.2) vis-à-vis (5.12c), (5.12d), etc. For intermediate values of A_A, Γ'_A, we may expect modest departures from the exact values above. Finally, using (5.11c) in (5.12) we can write alternatively

$$\langle \mathcal{E}^0 \rangle_A = 1 \; ; \quad \langle \mathcal{E}^2 \rangle_A = 1 \; ;$$

$$\langle \mathcal{E}^4 \rangle_A = 2 + \langle \hat{B}_{oA}^4 \rangle / A_A \langle \hat{B}_{oA}^2 \rangle^2 (1+\Gamma_A')^2 ;$$

$$\langle \mathcal{E}^6 \rangle_A = 6 + \frac{9\langle \hat{B}_{oA}^4 \rangle}{A_A \langle \hat{B}_{oA}^2 \rangle^2 (1+\Gamma_A')^2} + \frac{\langle \hat{B}_{oA}^6 \rangle}{A_A^2 \langle \hat{B}_{oA}^2 \rangle^3 (1+\Gamma_A')^3} \; ; \text{ etc.} \tag{5.13}$$

which shows how the (normalized moments behave as the Impulsive Index $A_A \to \infty$, or as the independent gaussian component becomes dominant ($\Gamma_A' \to \infty$).

For the odd moments of $\mathcal{E}$(or E) our procedure above, of course, is not applicable, and we must go directly to the calculation on the pdf, $w_1(\mathcal{E})_A$, cf. (5.1), (5.2), for k=1,3,5,... .

B. Exact Class B Even Moments:

The relations (5.10), (5.10a) apply here also, but the explicit differentiation of $F_1(ia\lambda)_B$, based on (2.51) cannot make direct use of a power series expansion of the integrand in the exponential, because the integral is now an improper integral $(0,\infty)$ which is not uniformly convergent (in λ) over the entire domain of integration [Courant, 1936,II Sec. 4, Chapter 4, p. 307 et seq.], so that term-wise expansion (in λ) of the integrand, as for Class A interference above, is not permitted. However, let us temporarily consider the case where $(z_o)_{max} < \infty$ (e.g., output signals of finite duration). Then, the term-wise expansion is permitted, as the integral is now both proper and uniformly convergent; (in fact, the resulting c.f. belongs formally to Class A). We proceed as in Class A above and next apply $\lim (z_o)_{max} \to \infty$ to the c.f., e.g. $F_1(ia\lambda)_B = \lim_{(z_o)_{max} \to \infty} F_1(ia\lambda | (z_o)_{max} < \infty)_{"B"}$, and hence to each term of (5.11), (5.12), etc., now specialized to the Class B parameters, A_B, Γ_B', $\hat{B}_{oB}$, etc. Thus, (5.11) applies again here, with $\Gamma_A' \to \Gamma_B'$, $A_A \to A_B$, $\hat{B}_{oA} \to \hat{B}_{oB}$, etc. We obtain the analogue of (5.12), for example:

$$\left.\begin{aligned}
&\left.\begin{aligned}\langle\varepsilon^0\rangle_B &= 1;\\ \langle\varepsilon^2\rangle_B &= 1,\end{aligned}\right\}\text{as expected;}\\
&\langle\varepsilon^4\rangle_B = \Omega_{4B}/\Omega_{2B}^2(1+\Gamma_B')^2 + 2;\\
&\langle\varepsilon^6\rangle_B = \Omega_{6B}/\Omega_{2B}^3(1+\Gamma_B')^3+9\Omega_{4B}/\Omega_{2B}^2(1+\Gamma_B')^2+6;\ \text{etc.}
\end{aligned}\right\}\ , \tag{5.14}$$

where specifically,

$$\Omega_{2k\text{-}B} \equiv \frac{A_B\left\langle\hat{B}_{oB}^{2k}\right\rangle}{2^k}\ [\underset{\sim}{\theta}_B' = A_o,\ e_{o\gamma},\ \mathcal{a}_{RT}]_B \quad ; \quad \langle\ \rangle_{\underset{\sim}{\theta}_B'} \equiv \left\langle\int_0^\infty(\ )dz\right\rangle_{\underset{\sim}{\theta}_B'}\ , \tag{5.14a}$$

cf. (2.87d) and $\hat{B}_{oB}$ in (2.87c). Again, for the odd-moments (k=1,3,5,...) we must use the approximate forms (5.4), (5.6), etc.

5.3 Remarks:

For Class B (and ∴ Class C) interference ($0 < \alpha < 2$), when (2.89) is used as an approximation for the c.f., it is clear that if we use (2.89) in (5.10a), then $\langle\varepsilon^2\rangle \to O(\lambda^{\alpha-2})_{\lambda=0}\to\infty$: the second moment does not exist. Of course, this divergence is simply the consequence of the inadequate approximation, a behaviour which is alleviated by the alternative approach using the results of Section 4, Eqs. (4.3)-(4.5) in (5.1), cf. (5.4)-(5.6).

Of perhaps greater interest is to note that, in terms of our general classification [Sec. (2.3) and Sec.(2.5-3)], Giordano and Haber [1970,1972], in effect, postulate a finite period of observation (0,T) for each member of the ensemble, e.g. Eq. (2.36) above is in force. This is equivalent to Class A operation, since itamounts to an abrupt truncation of the basic signal waveform $u_o(z)$ as emitted from the ARI stages of the typical narrow-band receiver, cf. Fig.(2.1)II. This, in turn, means that the receiver bandwidth is large enough vis-à-vis that of the input to pass it with negligible transients, a defining characteristic of Class A noise. Then all moments exist [cf. Sec. (5.1)] and the proper approximation for the

PD is (3.7b). This Class A, or truncated case, goes over into a Class B model as the observation period (0,T) becomes large vis-à-vis receiver bandwidth, e.g., as $T\Delta f_{ARI}>>1$. The type of approximate c.f. employed is then that given by (2.89), or more suitably, (2.90), which includes an independent gaussian component. These, as we have seen [cf. Sec. (3.2)] yield satisfactory approximations for small and intermediate ranges of envelope $\mathcal{E}$, or thresholds $\mathcal{E}_0$, but fail at some point ("large" $\mathcal{E}$, $\mathcal{E}_0$) to give the more rapid convergence needed to insure the physically required finite moments of all (positive) orders, cf. Sec. (5.1). Thus, the results of Giordano and Haber [1970,1972] (for suitably large $T\Delta f_{ARI}$), while practically useful as long as the statistics of very large values of the envelope are not demanded, are analytically incomplete as Class B models of the full range of possible values of the random envelope $\mathcal{E}$ and exceedance probability $P_1(\mathcal{E} > \mathcal{E}_0)_B$.

On the other hand, the important analysis of Furutsu and Ishida [1960], which represents a subclass of our Class B model in that a specific emitted waveform $[u_0(z)_B]$ is chosen, i.e. an exponential $\sim e^{-\alpha z}$, and several, particular spatial distributions of sources with a given propagation law ($\sim 1/\lambda$) are assumed, along with an exponential distribution of input signal amplitudes, does yield analytic forms of the PD and pdf which permit the existence of all orders of envelope moments, and which conform closely to the statistics of the atmospheric data studied therein. The approach of Furutsu and Ishida [1960] is similar to ours in that approximate c.f.'s are obtained, suitable for the small and large values of $\mathcal{E}$(and $\mathcal{E}_0$), while the intermediate ranges (of $\mathcal{E}$) are evaluated by numerical techniques. The canonical methods of our approach, however, are not invoked. Of course, neither Furutsu and Ishida, nor Giordano, and others, consider or distinguish Class A interference, which is a new category, as far as its statistical-physical description is concerned, considered originally by the author [Middleton, 1973, 1974].